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Phenomenological description of the optical field chaos in storage ring free-electron lasers

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The problem of optical field chaos in a storage ring free-electron laser oscillator has been discussed by using a phenomenological model. The result of theoretical analysis and numerical simulation shows that the variation of laser intensity versus time can be both periodic and chaotic when there is a weak gain modulation in the optical cavity. As time increases, the leading Lyapunov characteristic exponent of the system goes to a negative and a positive real number, respectively. Further research is carried out, and a chaotic transition via period-doubling bifurcation has been found when there are variations in the modulation parameters. The behavior of the system with two different kinds of gain modulation is also discussed.

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I. INTRODUCTION

Free-electron laser (FEL) physics has become an active area of research since the first FEL was operated at Stanford in 1976 [1]. As a result of the interaction between the relativistic electron beam and the electromagnetic fields, FEL comes to be a highly nonlinear system in which instabilities and chaos can easily be found. In recent years, along with the nonlinear study of the FEL, papers on the chaotic phenomena of the FELs have been accumulated [2–8]. In 1988, Riyopoulos and Tang studied the chaotic motion of electrons in a combined field of the wiggler magnetic field, the fundamental mode, and the sidebands [2]. Two years later, Chen and Davidson found that the trajectory of an electron in a helical wiggler with an axial guide field could also be chaotic when the electrostatic and magnetic self-fields of the electron beam could not be neglected [3]. At the same year, Spindler and Renz gave a detailed description of the chaotic behavior of electrons in a realizable wiggler field [4]. Even in a linearly polarized wiggler with an axial guide field, electrons can also move along chaotic orbits [5]. From these results we can see that the chaotic motion of the electrons in FELs is universal. However the optical field chaos, which may cause a broadening of the laser spectrum, has rarely been considered [6–8].

Theoretical study of the optical field chaos in a storage ring free-electron laser oscillator (SRFEL) was first carried out by Billardon in 1990 [7]. In the early 1980s, it was commonly believed that the FEL output laser light should have a continuous temporal structure. While the experiment made on the accelerator at Orsay (known as the ACO SRFEL) shows that the laser naturally adopts a pulsed structure, rather than a continuous one [9,10]. By using a phenomenological model with a weak periodic gain modulation, Billardon found that the laser could be either periodic or chaotic, and it was in good agreement with the experimental results [7,8].

In this present paper we will first simplify the equations given by Billardon, and then discuss the behavior of the simplified equations in detail. From the results of the numerical simulation one can see clearly that the optical field of the SRFEL can be either periodic or chaotic. This explains why the laser naturally has a pulsed temporal structure. We can also confirm the existence of the optical field chaos by calculating the Lyapunov characteristic exponents. Finally, the behavior of the system with two different kinds of gain modulation is also discussed.

II. PHENOMENOLOGICAL MODEL

The following model of a SRFEL system is considered:

$$\frac{dI}{dt}=I\frac{g-p}{\theta}\ , \tag{1}$$

$$\frac{d(\sigma^2)}{dt}=-\frac{2}{\tau_s}(\sigma^2-\sigma_0^2)+\alpha I\ , \tag{2}$$

$$g=g_0\exp[-k(\sigma^2-\sigma_0^2)][1+F(t)]\ , \tag{3}$$

where the first equation describes the amplification of the laser intensity I. The parameters g, p, and θ stand for the unsaturated gain, cavity losses, and the electron revolution period (which equals to the round-trip transit time of the light pulse stored inside the optical cavity), respectively. The second equation describes the evolution of the electron energy spread $\sigma=\Delta E/E_0$, where E_0 is the average energy of electrons and ΔE is the rms deviation. σ_0 is the equilibrium value of the energy spread without FEL operation. $\tau_s/2$ is the characteristic damping time of $\sigma^2-\sigma_0^2$ evolving without FEL operation, and it is usually of hundreds of milliseconds long. The term αI, where $\alpha>0$, represents an energy spread caused by the interaction between the optical field and the electron bunch. The third equation describes the evolution of the optical gain in the cavity, and $F(t)$ represents the modulation of the gain. As pointed out by Elleaume in Ref.

[9], the gain of a SRFEL can be modulated by several means: (a) Misalignment of the electron beam with the optical cavity axis may cause a transverse translation of the electron beam, and this results in a modulation of the gain. For ACO SRFEL, a 0.5 mm displacement is sufficient to decrease the gain by a factor of 4. (b) Altering the storage ring revolution frequency of the ACO SRFEL can shift the synchronism of the electron and the optical pulses in the cavity, and cause a modulation of the gain. From the ACO SRFEL experiment, 100 Hz of frequency changing is enough to stop the laser. (c) Modulation of the cavity length via a piezoelectric transducer can also cause a modulation of the gain. A 50 μm displacement can stop the laser. The first two methods have already been used on ACO SRFEL to get a stable periodic laser operation.

The model described by Eqs. (1)–(3) are simply phenomenological. They are similar to the rate equations of lasers utilizing stimulated emission between two different states of an atom or a molecular. For both g and α are functions of σ^2 and t, the system described by Eqs. (1)–(3) is a nonautonomous and nonlinear one. Thus solutions for this system can be either periodic or chaotic. In order to simplify Eqs. (1)–(3), we first assume g_0, p and θ are constants, and then set $\Sigma=\sigma^2-\sigma_0^2$. Now we can expand g and α near to the equilibrium point $(I,\Sigma)=(I_0,\Sigma_0)$, where $dI/dt|_{(I_0,\Sigma_0)}=0$, and $d\Sigma/dt|_{(I_0,\Sigma_0)}=0$, of Eqs. (1)–(3) when $F(t)=0$. Keeping just the linear term of the Taylor expansion we get

$$g(\Sigma,t)=p[1-k(\Sigma-\Sigma_0)][1+F(t)] , \tag{4}$$

$$\alpha(\Sigma)=\alpha(\Sigma_0) . \tag{5}$$

Substituting Eqs. (4) and (5) into Eqs. (1)–(3), we finally obtain a simplified form of Eqs. (1)–(3):

$$\frac{d\hat{I}}{dt}=\frac{1}{\tau_0}\hat{I}[1-\hat{\Sigma}+(1+\beta-\hat{\Sigma})F(t)] , \tag{6}$$

$$\frac{d\hat{\Sigma}}{dt}=\frac{2}{\tau_s}(\hat{I}-\hat{\Sigma}) , \tag{7}$$

where $\tau_0=\theta/(g_0-p)$, $\beta=1/k\Sigma_0$, $\hat{I}=\tau_s\alpha(\Sigma_0)I/2\Sigma_0$, and $\hat{\Sigma}=\Sigma/\Sigma_0$.

III. cw LASER OPERATION OF SRFEL

If the modulating term $F(t)$ is omitted, a cw laser operation is achieved. Now Eqs. (6) and (7) are reduced to a two-dimensional autonomous system:

$$\frac{d\hat{I}}{dt}=\frac{1}{\tau_0}\hat{I}(1-\hat{\Sigma}) , \tag{8}$$

$$\frac{d\hat{\Sigma}}{dt}=\frac{2}{\tau_s}(\hat{I}-\hat{\Sigma}) . \tag{9}$$

Equations (8) and (9) are the same as the correspondent equations in Ref. [9]. There are two fixed points of Eqs. (8) and (9) in the phase space $(\hat{I},\hat{\Sigma})$. They are (0,0) and (1,1). In the vicinity of point (0,0), the Jacobian matrix J of the linearized equations of (8) and (9) has two oppositely signed eigenvalues $1/\tau_0$ and $-2/\tau_s$. Therefore (0,0) is a saddle point. While in the vicinity of point (1,1), the two eigenvalues of J are complex conjugates with negative real part $(\tau_s \gg \tau_0)$: $-[1\pm i(2\tau_s/\tau_0-1)^{1/2}]/\tau_s$, so (1,1) is a stable focus.

Eliminating the variable $\hat{\Sigma}$ from Eqs. (8) and (9), we can get a second-order ordinary differential equation of $\hat{I}$. Setting $x=\tau_0\ln\hat{I}$, we have

$$\frac{d^2x}{dt^2}+\frac{2}{\tau_s}\frac{dx}{dt}+\frac{2}{\tau_s}(e^{x/\tau_0}-1)=0 . \tag{10}$$

This is an equation of a damped nonlinear oscillator. For (1,1) is a stable focus, it is very clear that the behavior of the solution to Eq. (10) near point (1,1) is the same as a damped linear oscillator. An approximate solution to Eqs. (8) and (9) near (1,1) can be obtained analytically, by expanding $\hat{I}$ and $\hat{\Sigma}$ around point (1,1):

$$\hat{I}=1+\delta\hat{I} , \tag{11}$$

$$\hat{\Sigma}=1+\delta\hat{\Sigma} . \tag{12}$$

Substituting Eqs. (11) and (12) into Eqs. (8) and (9) yields

$$\frac{d^2\delta\hat{I}}{dt^2}+\frac{2}{\tau_s}\frac{d\delta\hat{I}}{dt}+\frac{2}{\tau_0\tau_s}\delta\hat{I}=0 . \tag{13}$$

This is an equation of a damped linear oscillator. Its solution can be easily written down:

$$\delta\hat{I}=\delta\hat{I}_0e^{-(t-t_0)/\tau_s}\cos[\Omega_R(t-t_0)] , \tag{14}$$

where $\Omega_R=(2/\tau_0\tau_s-1/\tau_s^2)^{1/2}$ is the circular frequency.

Numerical simulating results are in good agreement with the theoretical results. In Fig. 1, the phase space trajectory of Eqs. (8) and (9) is shown. The initial point of this trajectory is $(\hat{I},\hat{\Sigma})=(10^{-5},0)$, and the parameters in Eqs. (8) and (9) are taken as $\tau_0=5$ ms and $\tau_s=200$ ms. Further investigation shows that whenever g and α are independent of t, Eqs. (6) and (7) will be autonomous, and we will have a cw laser operation.

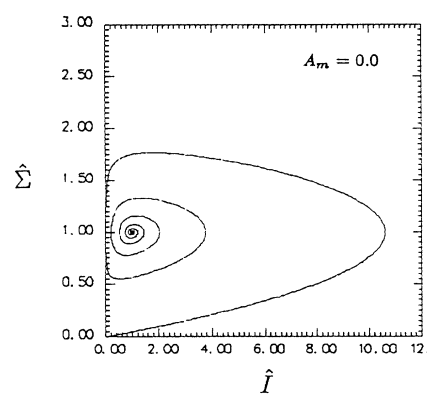

FIG. 1. Phase space diagram of a cw laser operation of SRFEL. $F(t)=0$. Parameters are taken as $\tau_0=5$ ms, $\tau_s=200$ ms.

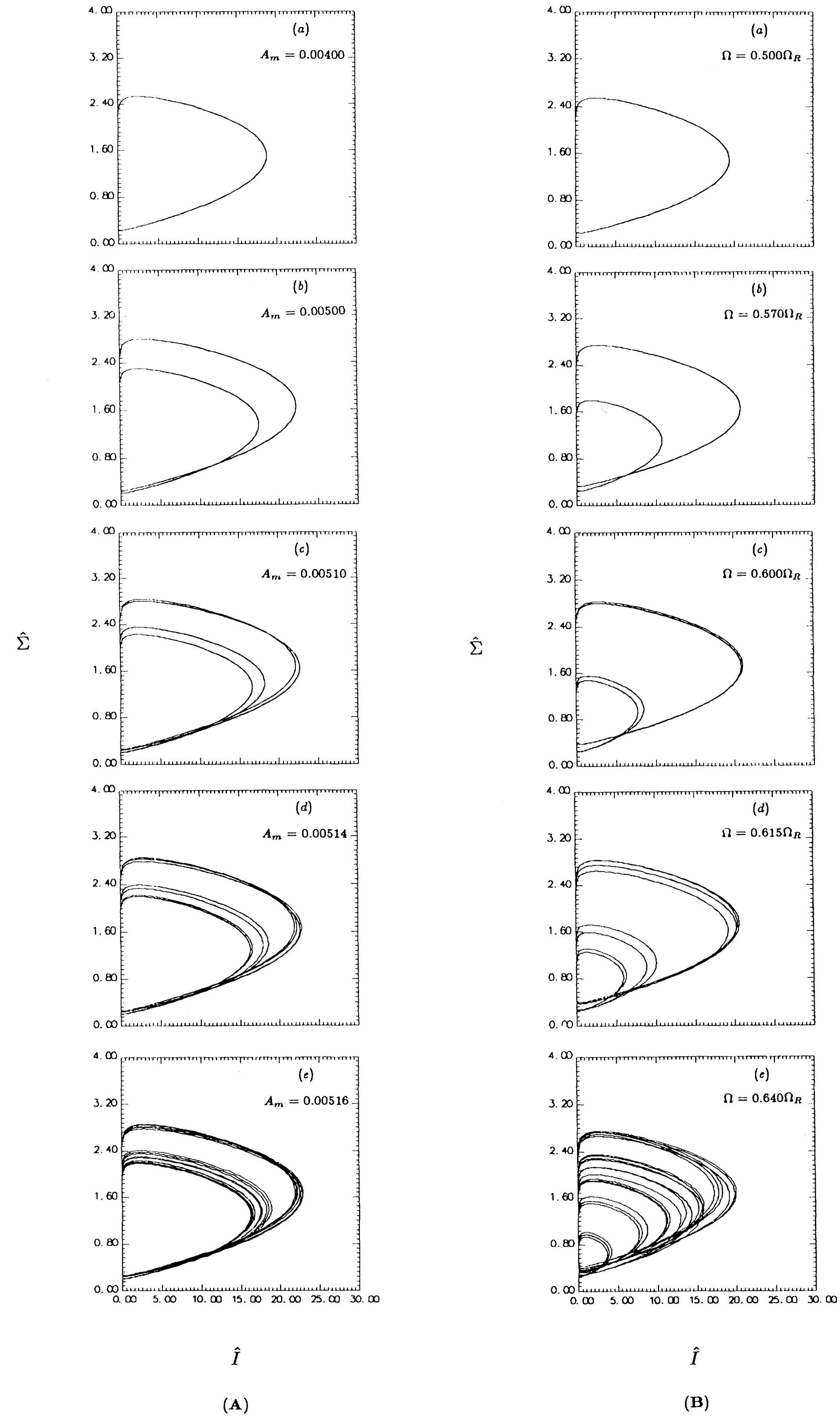

FIG. 2. Phase space diagrams of a pulsed laser operation of SRFEL. $F(t)=A_m\sin(\Omega t)$, $\tau_0=5$ ms, $\tau_s=200$ ms, $\beta=135.63$. (A) $\Omega=\Omega_R$. (a) $A_m=0.004\,00$, (b) $A_m=0.005\,00$, (c) $A_m=0.005\,10$, (d) $A_m=0.005\,14$, (e) $A_m=0.005\,16$. (B) $A_m=0.005\,50$. (a) $\Omega=0.500\Omega_R$, (b) $\Omega=0.570\Omega_R$, (c) $\Omega=0.600\Omega_R$, (d) $\Omega=0.615\Omega_R$, (e) $\Omega=0.640\Omega_R$.

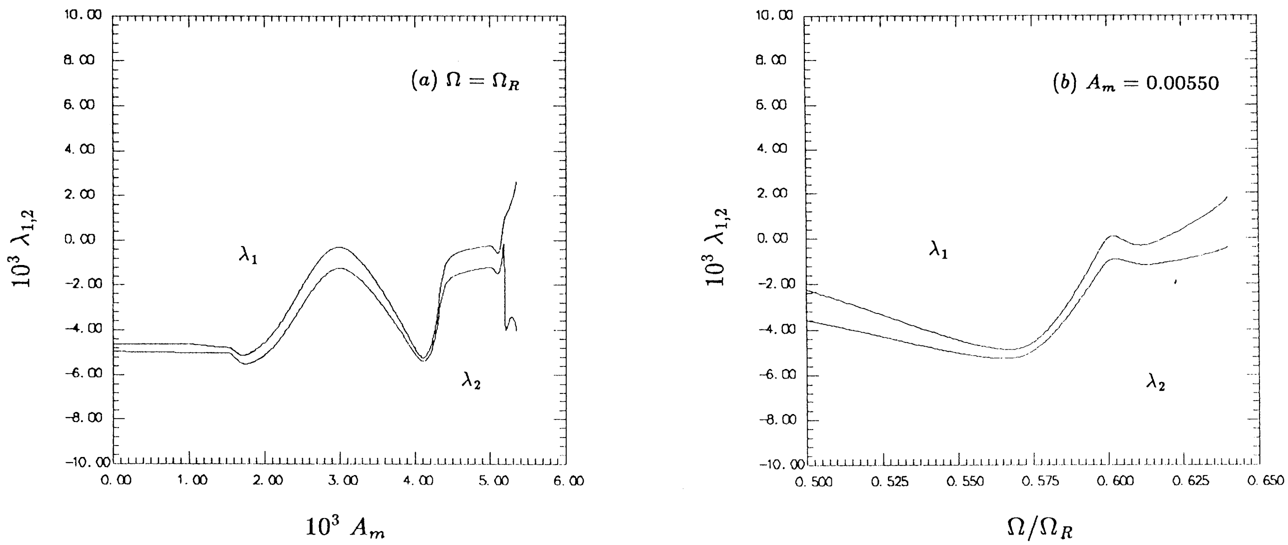

FIG. 3 (a) Lyapunov characteristic exponents λ_1 and λ_2 versus A_m. $\Omega=\Omega_R$. (b) λ_1 and λ_2 versus Ω/Ω_R. $A_m=0.005\,50$. Parameters τ_0, τ_s, and β are the same as those in Fig. 2.

IV. PULSED LASER OPERATION OF SRFEL WITH $F(t)=A_m\sin(\Omega t)$

If the modulating term $F(t)$ is considered, Eqs. (6) and (7) will be nonautonomous. In this section we will study a SRFEL with a weak periodic gain modulation

$$F(t)=A_m\sin(\Omega t)\ . \tag{15}$$

When A_m is small enough, expansion Eqs. (11) and (12) are still valid. Substituting Eqs. (11) and (12) into Eqs. (6) and (7) yields

$$\frac{d^2\delta\hat{\Sigma}}{dt^2}+\frac{2}{\tau_s}\frac{d\delta\hat{\Sigma}}{dt}+\frac{2}{\tau_0\tau_s}\delta\hat{\Sigma}=\frac{2\beta A_m}{\tau_0\tau_s}\sin(\Omega t)\ . \tag{16}$$

This is an equation of a forced linear oscillator, where a damping term is included. Its solution is

$$\delta\hat{\Sigma}=\delta\hat{\Sigma}_0 e^{-(t-t_0)/\tau_s}\cos[\Omega_R(t-t_0)] +\frac{\beta A_m}{\tau_0[(1/\tau_0-\tau_s\Omega^2/2)^2+\Omega^2]^{1/2}}\sin(\Omega t-\varphi)\ , \tag{17}$$

where $\varphi=\tan^{-1}[\Omega/(1/\tau_0-\tau_s\Omega^2/2)]$. When $\Omega=(2/\tau_0\tau_s-2/\tau_s^2)^{1/2}\approx\Omega_R$, the second term on the right-hand side of Eq. (17) comes to its maximum value $\beta A_m/\tau_0\Omega_R$. This is the usual resonance. From Eq. (17) we can see that the laser becomes a pulsed one when t is big.

When A_m is not very small, expansion Eqs. (11) and (12) cannot be used. In fact, Eqs. (6) and (7) are similar to the equations of the forced Brusselator [11]. Integrating Eqs. (6) and (7) numerically along line $\Omega=\Omega_R$ and line $A_m=0.005\,50$ in the parameter space (Ω, A_m), we found there is a chaotic transition via period-doubling bifurcation when A_m or Ω changes, respectively. Phase space diagrams of solutions to Eqs. (6) and (7) of different periods are shown in Fig. 2.

In order to confirm the existence of the optical field chaos, the Lyapunov characteristic exponents are calculated [12,13]. For a periodic orbit in the phase space, the two Lyapunov characteristic exponents λ_1 and λ_2 (where $\lambda_1>\lambda_2$) should be all negative. While for a chaotic orbit, one of them λ_1 will be positive. Figure 3 shows the variation of λ_1 and λ_2 versus A_m and Ω. It is very clear that for some values of A_m and Ω, λ_1 is positive. This indicates the existence of the optical field chaos.

Further numerical study is carried out. In the parameter space (Ω, A_m), we can see a more complicated structure of bifurcation and chaos, as shown in Fig. 4. This result explains the complex behavior of the output laser observed in the ACO SRFEL experiment.

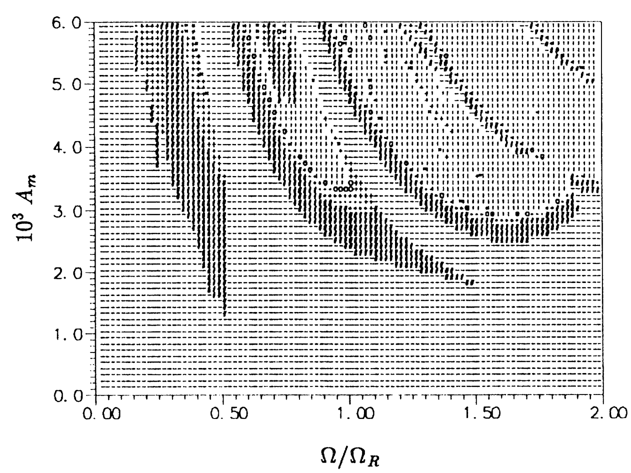

FIG. 4. Structure of bifurcation and chaos in the parameter space (Ω, A_m). Parameters τ_0, τ_s, and β are the same as those in Fig. 2. Symbols "–", "#", "*", "o", "+", "×", and "|" represent period 1, 2, 4, 8, 3, 6, and chaotic orbit, respectively.

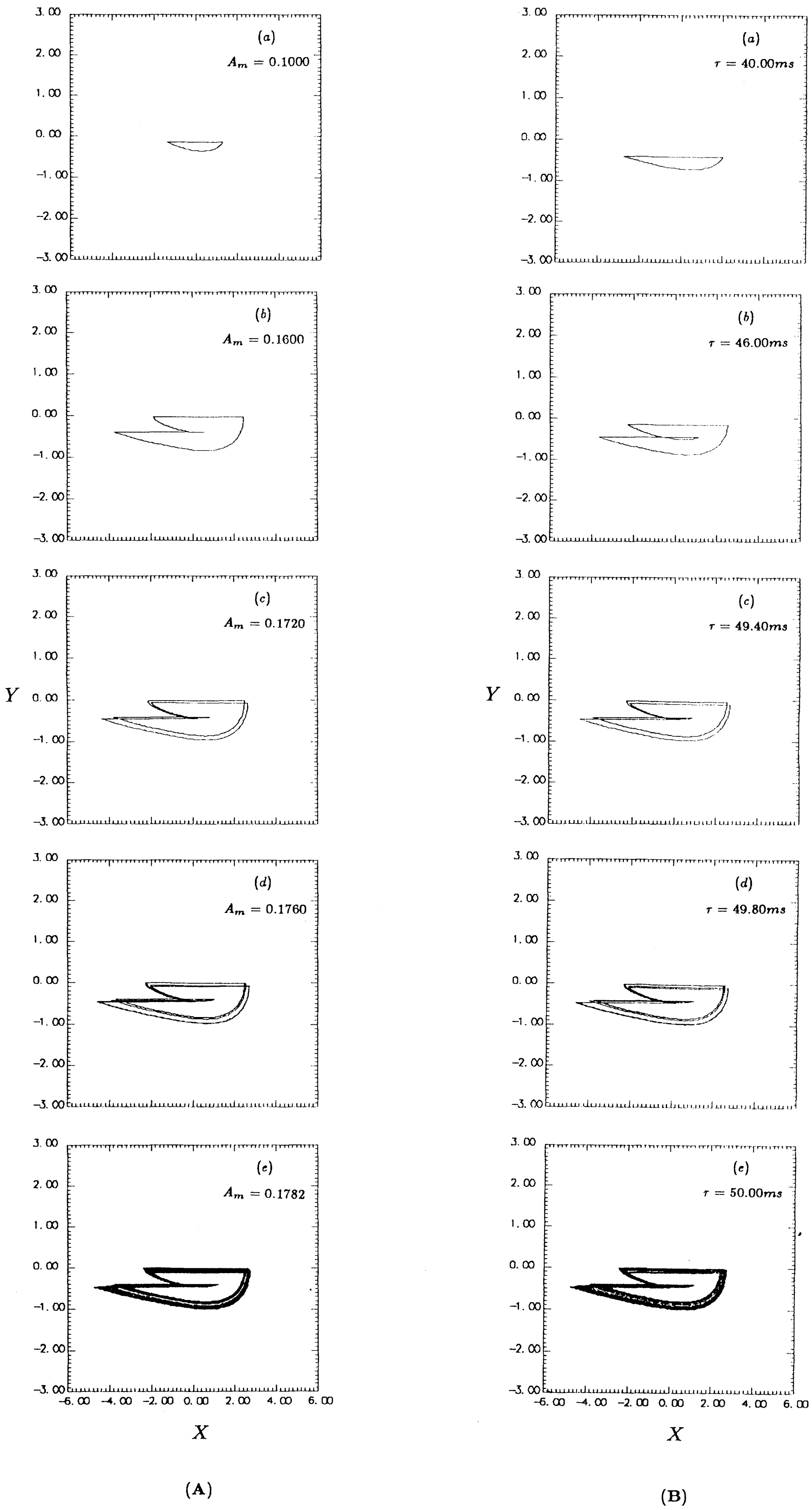

FIG. 5. Phase space diagrams of a pulsed laser operation of SRFEL. $F(t)=A_m\sum_{n=1}^{\infty}\delta(t-n\tau)$. $\tau_0=5$ ms, $\tau_s=200$ ms, $\beta=135.63$. (A) $\tau=50.00$ ms. (a) $A_m=0.1000$, (b) $A_m=0.1600$, (c) $A_m=0.1720$, (d) $A_m=0.1760$, (e) $A_m=0.1782$. (B) $A_m=0.1783$. (a) $\tau=40.00$ ms, (b) $\tau=46.00$ ms, (c) $\tau=49.40$ ms, (4) $\tau=49.80$ ms, (e) $\tau=50.00$ ms.

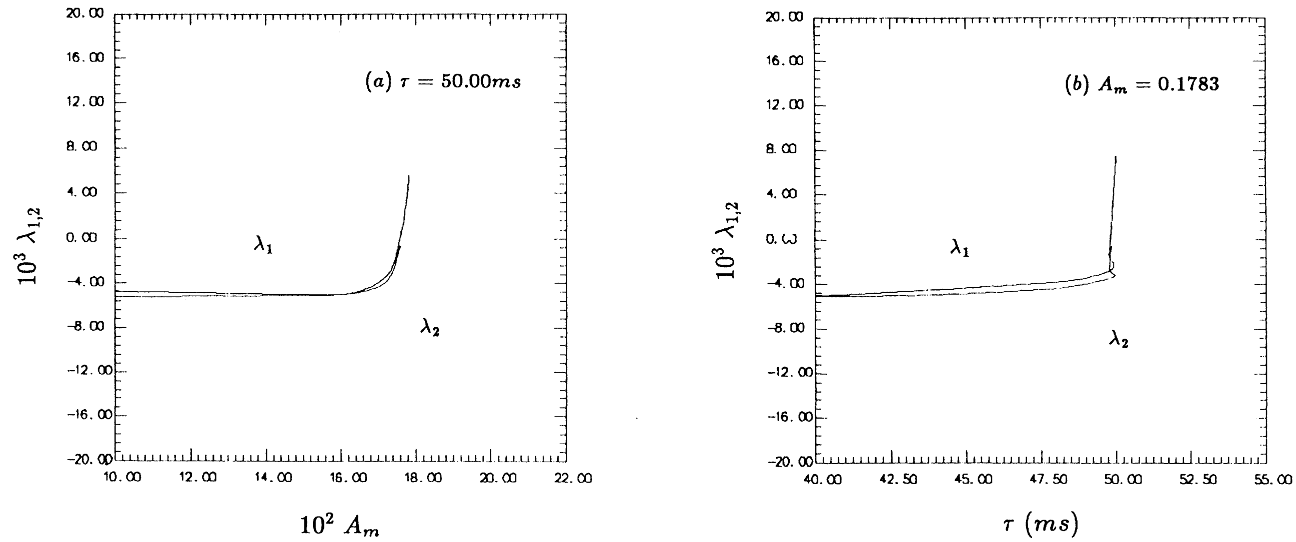

FIG. 6. (a) Lyapunov characteristic exponents λ_1 and λ_2 versus A_m. $\tau=50.00$ ms. (b) λ_1 and λ_2 versus τ. $A_m=0.1783$. Parameters τ_0, τ_s, and β are the same as those in Fig. 5.

V. PULSED LASER OPERATION OF SRFEL WITH $F(t)=A_m\sum_{n=1}^{\infty}\delta(t-n\tau)$

In this section we will study a SRFEL with a pulsed periodic gain modulation

$$F(t)=A_m\sum_{n=1}^{\infty}\delta(t-n\tau)\ , \tag{18}$$

where τ is the interval of the modulating pulses. If the gain modulation in the optical cavity is not a continuous *sine* wave one, but a pulsed one described by Eq. (18), we can still solve Eqs. (6) and (7) numerically. Before doing this, we first simplify Eqs. (6) and (7) by utilizing the following transformation:

$$x=\ln\hat{I}\ , \tag{19}$$

$$y=1-\hat{\Sigma}\ . \tag{20}$$

Now Eqs. (6) and (7) can be rewritten as

$$\frac{dx}{dt}=\frac{1}{\tau_0}[y+(\beta+y)F(t)]\ , \tag{21}$$

$$\frac{dy}{dt}=-\frac{2}{\tau_s}(y+e^x-1)\ . \tag{22}$$

As in Sec. IV, we still integrate Eqs. (21) and (22) along line $\tau=50.00$ ms and line $A_m=0.1783$ in the parameter space (τ, A_m). Numerical simulating results shows that there is again a chaotic transition via period-doubling bifurcation when the correspondent A_m or τ changes. The phase space diagrams of different periods and chaos along line $\tau=50.00$ ms and line $A_m=0.1783$ are displayed in Fig. 5. As a second step, we carried out the calculation of the Lyapunov characteristic exponents λ_1 and λ_2 for different parameters τ and A_m. Figure 6 shows λ_1 and λ_2 versus τ and A_m. From the emergence of positive λ_1 we can confirm the existence of the optical field chaos. The structure of bifurcation and chaos in the parameter space (τ, A_m) is also very complicated. This can be easily seen from Fig. 7.

VI. COMPARISON OF EXPERIMENTAL FACTS AND NUMERICAL SIMULATION RESULTS

As we have pointed out above, the ACO SRFEL experiment shows that when there is a gain modulation in the optical cavity, the output laser temporal structure can be either periodic or chaotic, as shown in Fig. 8(A). The different temporal structure of the output laser pulses are determined by the amount of the modulating parameters.

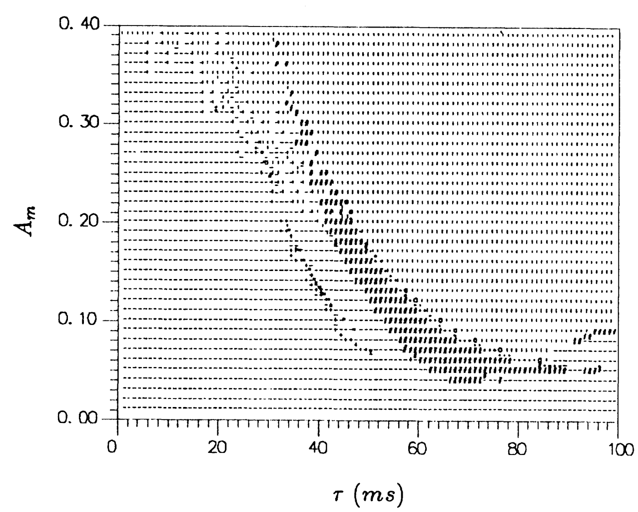

FIG. 7. Structure of bifurcation and chaos in the parameter space (τ, A_m). Parameters τ_0, τ_s, and β are the same as those in Fig. 5. Symbols "−", "#", "*", "o", "+", "×", and "|" represent period 1, 2, 4, 8, 3, 6, and chaotic orbit, respectively.

Using the present phenomenological model, we can get a solution of laser intensity $\hat{I}$ as a function of time t. When the modulating term is taken as $F(t)=A_m\sin(\Omega t)$, the solutions of $\hat{I}(t)$ will be determined by the values of A_m and Ω. Periodic and chaotic solutions have been obtained, as shown in Fig. 8(B). When $F(t)=A_m\sum_{n=1}^{\infty}\delta(t-n\tau)$, we have also gotten a similar solution of $\hat{I}(t)$, which is shown in Fig. 8(C). From these results we can see that a phenomenological SRFEL model with a gain modulation can really have a pulsed solution of the laser intensity $\hat{I}$ of different periods. This is in good agreement with the experimental results. We have also found that the shape of the laser pulses obtained here is a little different when $F(t)$ is of different forms. According to Fig. 8 we can say roughly that a pulsed gain modulation described by Eq. (18) conforms better to experimental facts than a *sine* wave one given by Eq. (15) does.

VII. CONCLUSIONS

As a summary, we would like to review the principal points of this paper. By using a phenomenological model, the problem of optical field chaos in a SRFEL has been discussed. Adopting a Taylor expansion, we first simplified the original equations of the laser intensity I and the electron energy spread σ. This makes it simple to analyze the behavior of the system. Theoretical study and numerical simulation have been carried out, and chaotic transitions via period-doubling bifurcation have been found when there is a gain modulation in the optical cavity. The appearance of the optical field chaos has been confirmed by calculating the Lyapunov characteristic exponents of the system. Numerical results also indicate that the existence of a gain modulation leads to a pulsed laser operation. For both a continuous *sine* wave gain

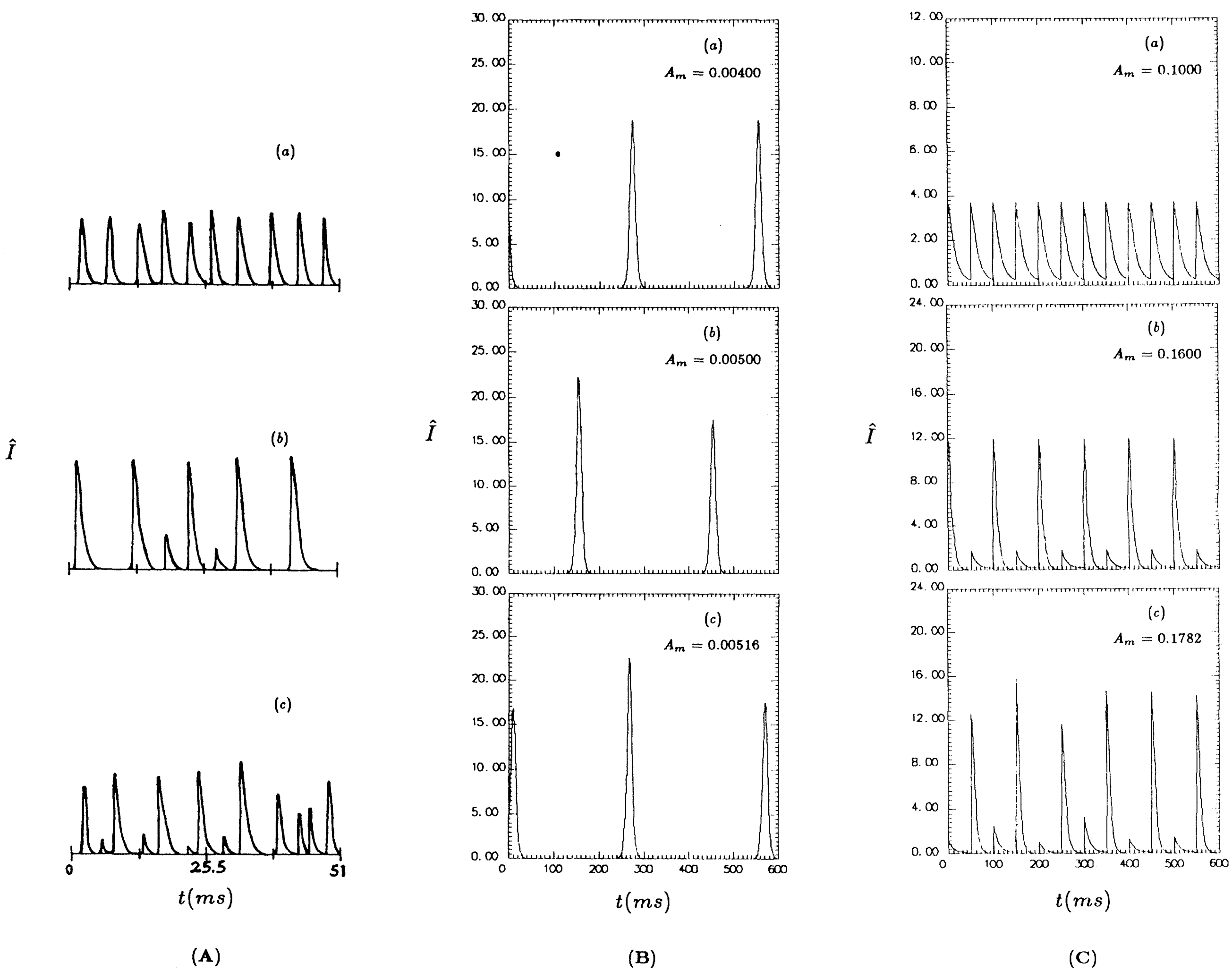

FIG. 8. Comparison of the experimental results and the numerical simulating results. (A) output laser temporal structure observed in the ACO SRFEL experiment [7]. (B) Numerical results of the laser intensity $\hat{I}$ versus time t when the gain modulation is $F(t)=A_m\sin(\Omega t)$. (C) Numerical results of the laser intensity $\hat{I}$ versus time t when the gain modulation is $F(t)=A_m\sum_{n=1}^{\infty}\delta(t-n\tau)$.

modulation term and a pulsed one, the system expresses similar features. Further study shows that the structure of bifurcation and chaos in the parameter space is very complicated. This explains why the output laser of the ACO SRFEL experiment is usually of a complex pulsed feature, not a continuous wave one. Comparing with the experimental facts, we can say roughly that the gain modulation in the optical cavity seems to be a pulsed one, rather than a *sine* one.

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[1] L. R. Elias, W. M. Fairband, J. M. J. Madey, H. A. Schwettman, and T. I. Smith, Phys. Rev. Lett. **36**, 710 (1976).
[2] S. Riyopoulos and C. M. Tang, Phys. Fluids **31**, 3387 (1988).
[3] C. Chen and R. C. Davidson, Phys. Rev. A **43**, 5541 (1991).
[4] G. Spindler and G. Renz, Phys. Fluids B **3**, 3517 (1991).
[5] L. Michel, A. Bourdier, and J. M. Buzzi, Phys. Fluids B **5**, 965 (1993).
[6] S. J. Hahn and J. K. Lee, Phys. Lett. A **176**, 339 (1993).
[7] M. Billardon, Phys. Rev. Lett. **65**, 713 (1990).
[8] M. Billardon, Nucl. Instrum. Methods A **304**, 37 (1991).
[9] P. Elleaume, J. Phys. (Paris), **45**, 997 (1984).
[10] M. Billardon, P. Elleaume, J. M. Ortega, C. Bazin, M. Bergher, M. Velghe, D. A. G. Deacon, and Y. Petroff, IEEE J. Quantum Electron. **QE-21**, 805 (1985).
[11] G. R. Wang, Acta Phys. Sin. Abstr. **32**, 960 (1983) (in Chinese).
[12] A. Wolf, B. Swift, H. L. Swinney, and J. A. Vastano, Physica D **16**, 285 (1985).
[13] S. N. Rasband, *Chaotic Dynamics of Nonlinear Systems* (Wiley, New York, 1990).